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PHENOMENOLOGICAL DESCRIPTION OF THE AXISYMMETRIC AIR-BUBBLE PLUME

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Abstract—The axisymmetric air-bubble plume in water may be thought of as emanating from a virtual point source located at some distance below the real source. To give a phenomenological description of this system one has at one's disposal the equation of continuity, the balance equation for momentum, and the balance equation for kinetic energy. The most common way of approach so far has been to make use of the first two of these equations, together with the assumption that the rate of entrainment be proportional to the vertical centerline velocity. This is the theory developed, in particular, by Ditmars & Cederwall (1974). The present paper presents an alternative theory, in which the rate-of-entrainment assumption is abandoned and use is made of the kinetic energy equation together with the assumption that the most dominant component of the Reynolds stress be self-preserved. Very good agreement is found in this way with the large scale experiments of Kobus (1968) and Milgram (1983). The agreement is somewhat better than that found for the *plane* plume; the analogous theory for this case was developed by one of the present authors (I.B.). The conclusion of our analysis is that the kinetic energy approach stands out as a quite viable alternative for engineering applications in the axisymmetric case. Copyright © 1996 Elsevier Science Ltd.

Key Words: two-phase flow, air-bubble plumes, turbulence

1. INTRODUCTION

From an engineering viewpoint the air-bubble plume—whether it is two-dimensional emanating from orifices in a pipeline or axisymmetric emanating from a (virtual) point source—is a useful device for a number of reasons. Some of them are:

(1) Production of an opposing surface current intended to damp or even kinematically stop incoming waves from the sea, thus protecting harbour areas from damage. The theory of Taylor (1955) based upon prior model scale experiments, and the subsequent large scale experiments of Bulson (1961, 1963, 1968), are important works in this direction (a generalization of Taylor's theory was later given by Brevik 1976).
(2) Maintenance of ice-free conditions during the winter season. Installations of this kind are in current use in Norwegian fjords.
(3) Creation of good mixing conditions in more or less isolated fjords. It happens that the entrance region connecting the fjord with the open sea on the outside is so shallow and narrow that the tidal currents are insufficient to cleanse the fjord. Installation of an air curtain emanating from a pipeline situated at some convenient depth has in practice turned out to be quite an efficient remedy to bring oxygen-poor dirty water up to the free surface where the pollutants evaporate. Again, installations of this type are in use in Norwegian fjords. Destratification of drinking water reservoirs is another important application (see, for instance, Schladow 1992).
(4) Creation of "air-lift" in an oil reservoir in cases when the natural pressure in the reservoir is diminishing, thus helping one to improve the efficiency during oil exploration.

There are basically two different versions of an air-bubble system. Either, the plume is two-dimensional (plane) on a large scale, created by air emanating through orifices in a horizontal pipeline. Or, the plume is axisymmetric, emanating from a single (virtual) point source. In any case, a plume is a turbulent and very complicated system when looked upon in detail. The degree of complexity met with when attempting to describe such a plume theoretically, depends evidently on which *scale* is chosen to be important. There are numerous works in the literature discussing the behaviour of bubble plumes, from various points of view. We may mention review papers of van Wijngaarden (1972), Harper (1972), Miksis & Ting (1992), the book of Clift *et al.* (1978), and most recently, the book of Brennen (1995). There exist several doctor theses, among them Mäder (1971), Goossens (1979), Sjøen (1983), Johansen (1990), Torvik (1990) and Jakobsen (1993), all of which contain a considerable amount of general material about bubble plumes. The (unpublished) report of Haaland (1979) gives an extensive and useful critical survey over the applicability of different mathematical models for plumes. As regards ordinary research papers, we mention the ones of McDougall (1978) and Asaeda & Imberger (1993), both dealing with plumes in stratified environments. Kumar & Brennen (1993) consider nonlinear interactive effects between bubbles. Islam *et al.* (1995) propose a model in which the spreading rate for an axisymmetric plume is taken to be exponential.

In engineering applications it is usually sufficient to work on the gross scale level, implying that the air-water mixture may be considered as a continuous fluid whose density ρ is slightly varying throughout the plume. Early work on the equations of motion for an air-water mixture was made by van Wijngaarden (1968). In engineering practice one usually considers three basic equations, *viz.* the continuity equation, the balance equation for momentum, and the balance equation for kinetic energy. Also, some sort of closure relation is needed. Most workers in the field follow Ditmars & Cederwall (1974); cf. also Wilkinson (1979), in using only the two first of these equations, adopting as closure conditions that the rate of entrainment be proportional to the centerline velocity. As a possible alternative approach, one of us (Brevik 1977) developed for the case of a *plane* plume theory in which the kinetic energy balance equation was invoked, together with the closure condition that the most dominant Reynolds stress component be self-preserved. The theory was compared with the extensive large scale experiments of Kobus (1968, 1970, 1972). The main result of the comparison was that the measured standard deviation σ for the mean vertical water velocity was reproduced quite well by the theory, whereas the theoretical values for the centerline velocity u_c came out somewhat high. Moreover, because of the experiment, we adopted the following value for the relative bubble velocity u_{rel} with respect to the ambient water:

$$u_{rel} = 0.40 \text{ m/s}. \qquad [1]$$

This value is probably also somewhat high; when single bubbles rise through still water it is known from observations that the terminal velocity lies between 20 and 30 cm/s when the bubble diameter lies between 1 mm and 1 cm. Classic studies on single bubble rise are those of Haberman & Morton (1953) and Siemens (1954). A useful compilation of data is given in Fig. 7.3 in Clift *et al.* (1978). We note here that because of the strong turbulence in the plume, this case is physically quite different from that of single bubble rise. There is interaction between the bubbles influencing their collective terminal velocity (cf. figure 11 in Kobus (1968), and also the discussion in section 6.1 below). It would be premature simply to reject [1] on the basis of single bubble dynamics. However, when all aspects of this kinematic energy theory are taken together, we think that it is fair to conclude that its practical usefulness, in the case of two-dimensional plumes, seems to be limited.

The purpose of the present paper is to develop the same kind of theory, involving use of the kinetic energy equation together with the assumption about self-preservation for the most dominant Reynolds stress component, for the axisymmetric bubble plume. Remarkably enough, it turns out that the agreement with Kobus' experiments is now quite good, much better than it was in the plane case. Thus *both* the standard deviation σ *and* the centerline velocity u_c are reproduced fairly well. Moreover there is no longer any need of straining the value of the input parameter u_{rel}; we can simply adopt the value

$$u_{rel} = 0.30 \text{ m/s}, \qquad [2]$$

which is quite reasonable. These properties in general obviously support our confidence in the kinetic energy theory in the axisymmetric case. Also, the agreement with the more recent experiment of Milgram (1983) turns out to be quite good. An additional and unexpected result of the analysis is that the integral expressions for two quantities, called I_{plane} and I, turn out to be of almost the same magnitude in the two cases. These quantities are defined as follows.

Consider first the two-dimensional case. Assume stationary conditions, and let an overbar denote a mean value. For the velocity components u_k of the fluid we thus have $u_k = \bar{u}_k + u'_k$, where a prime denotes the fluctuating component. Let the positive z axis be pointing upwards, and let $u_c(z)$ be the mean centerline velocity. The central cross-correlation term for the velocity is $\overline{u'_x u'_z}$. We assume that the self-preservation property holds for the correlation term. This implies that

$$\frac{\overline{u'_x u'_z}}{u_c^2(z)} = f(\eta), \qquad [3]$$

in which f is an unspecified function of the parameter $\eta = x/\sigma(z)$, $\sigma(z)$ being the standard deviation for the velocity field. We can now write down the definition of I_{plane}:

$$I_{\text{plane}} \equiv \left(\frac{24}{\pi}\right)^{1/2} \int_0^\infty \eta f e^{(-1/2)\eta^2} d\eta = \text{constant}. \qquad [4]$$

The prefactor is included for practical reasons. In the theoretical solution obtained from the formalism, I_{plane} plays the role of an input parameter.

The definition of I in the axisymmetric case is quite analogous. We introduce cylindrical coordinates r, θ, z, the positive z axis pointing upwards as before. The most dominant correlation term is now $\overline{u'_r u'_z}$. Imposition of the self-preservation property yields

$$\frac{\overline{u'_r u'_z}}{u_c^2(z)} = f(\eta), \qquad [5]$$

where f is an unspecified function of the parameter $\eta = r/\sigma(z)$ (note that u_c, f, and σ are here quantities referring to the axisymmetric case only). The definition of I is

$$I \equiv 6\int_0^\infty \eta^2 f e^{(-1/2)\eta^2} d\eta = \text{constant}. \qquad [6]$$

The experimental result that the magnitudes of I_{plane} and I turn out to be almost the same in the two cases, hints towards some kind of universality property for this constant.

The general fact that it is easier to fit a simple phenomenological theory to the axisymmetric plume than to a two-dimensional plume, is perhaps not so surprising after all. An air bubble is a *three-dimensional* object. To picture the air bubble swarm from a horizontal pipeline as a two-dimensional bubble screen can only serve as a very rough approximation to the real situation, the more so because of the finite spacing between the orifices in the pipe. It might seem natural beforehand, on physical grounds, to expect that the neglect of azimuthal variation in a plume generated from a single source is a better approximation than the neglect of longitudinal variation in a plume generated from a pipeline.

2. ON THE TWO-DIMENSIONAL PLUME

For reference purposes we shall give a brief survey of the theory for a two-dimensional plume, as given in Brevik (1977). The situation is sketched in figure 1. The origin of the coordinate system is taken to lie at the bottom, the y axis lying along the pipe. The water depth is D, the atmospheric pressure as a head of water is P; the total head is $D^* = D + P$. The air-water mixture is regarded as a fluid with effective density $\rho(z, x)$, being slightly less than the density ρ_w of ambient water. We let ρ_a be the density of air in the bubbles, and m the mass of air in all bubbles per unit volume. The mean of the vertical component of the momentum balance equation can be written

$$\frac{\partial \bar{p}_d}{\partial z} + \rho_w \frac{\partial}{\partial x_k}(\bar{u}_z \bar{u}_k + \overline{u'_z u'_k}) = \bar{m} g \frac{\rho_w}{\rho_a}, \qquad [7]$$

where $\bar{p}_d$ is the mean of the dynamic pressure $p_d = p - \rho_w g(D - z)$, p being the total pressure. We shall henceforth neglect p_d. This is a common approximation made in engineering bubble plume theory. The approximation seems to be quite reasonable: under stationary conditions p_d should be expected to fluctuate around zero, or very close to this. We also assume that the fluctuations are

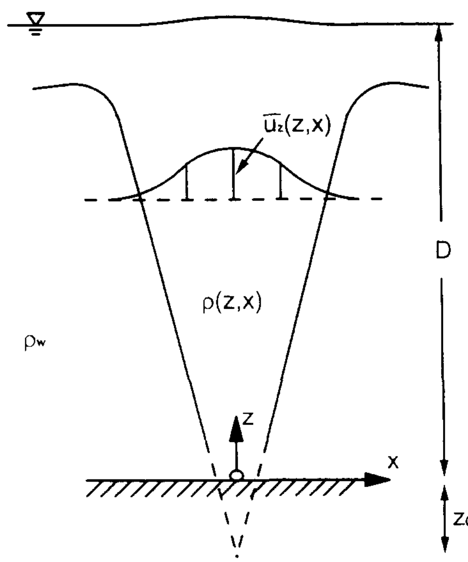

Figure 1. Sketch of the bubble plume. The figure shows the two-dimensional case.

small compared with the mean velocity in the vertical direction: $\overline{u_z'^2}/\bar{u}_z^2 \ll 1$, and we take the lateral variations of $\bar{u}_z$ and $\bar{m}$ to be given by Gaussian distributions:

$$\bar{u}_z(z, x) = u_c(z)\exp\left[-\frac{x^2}{2\sigma^2(z)}\right], \quad [8]$$

$$\bar{m}(z, x) = m_c(z)\exp\left[-\frac{x^2}{2\lambda^2\sigma^2(z)}\right]. \quad [9]$$

Here u_c, m_c are mean centerline quantities, and σ is the standard deviation for the velocity field. We shall treat λ as a constant. In reality this is only a rough approximation. The motivation for adopting this approximation is twofold: (i) it is mathematically simplifying; and (ii) the predicted behaviour of the plume is rather insensitive to different input values for λ. For these reasons it has been become common to put λ = const in the engineering literature. The proposed values of λ vary between 0.2 (Ditmars & Cederwall 1974) and 1 (Mäder 1971, Goossens 1979). Assuming isothermal expansion of the air as it rises, and taking into account the conservation equation for the mass of rising air, we arrive at the following differential equation for u_c and σ:

$$\frac{\mathrm{d}}{\mathrm{d}z}[u_c^2(z)\sigma(z)] = \left(\frac{1+\lambda^2}{\pi}\right)^{1/2}\frac{gQ^0P}{D^*-z}\frac{1}{u_z(z)+(1+\lambda^2)^{1/2}u_{\mathrm{rel}}}. \quad [10]$$

Here Q^0 is the expenditure of air per unit length of pipe, reduced to atmospheric pressure, and u_{rel} is the bubble-water slip velocity (assumed to be a constant, for simplifying reasons). Equation [10] was obtained also by Ditmars and Cederwall (1974).

The same technique can be applied to the kinetic energy balance equation. On the differential level this equation reads to leading order

$$\frac{\partial}{\partial x_k}\left(\frac{1}{2}\bar{u}_z^2\bar{u}_k\right) + \bar{u}_z\frac{\partial}{\partial x}(\overline{u_z'u_x'}) = \frac{\bar{m}\bar{u}_z g}{\rho_a}. \quad [11]$$

An important assumption in the theory is that of self-preservation. With regard to $\bar{u}_z$ and $\bar{m}$, the self-preservation property was built into the formalism of equations [8] and [9]. With regard to the cross-correlation term, this property was written down in [5]. Based upon these assumptions we derive the following energy equation:

$$\frac{\mathrm{d}}{\mathrm{d}z}[u_c^3(z)\sigma(z)] = \left(\frac{6}{\pi}\right)^{1/2}\frac{gQ^0P}{D^*-z}\frac{u_c(z)}{u_c(z)+(1+\lambda^2)^{1/2}u_{\mathrm{rel}}} - I_{\mathrm{plane}}u_c^3(z), \quad [12]$$

in which I_{plane} is as given in [4]. The value of I_{plane} has to be inferred from experiments. Using data from the large scale experiment of Kobus (1968), with $D = 4.30$ m, we obtain

$$I_{\text{plane}} = 0.13. \qquad [13]$$

As regards the value of λ, Brevik (1977) follows Ditmars & Cederwall (1974) in putting $\lambda = 0.2$ throughout. As already mentioned, u_{rel} is chosen to be 0.40 m/s in order to obtain rough agreement with Kobus' data.

3. AXISYMMETRIC PLUME: GOVERNING EQUATIONS

An analogous description will now be given for the axisymmetric plume. As already mentioned we introduce cylindrical coordinates r, θ, z, and let for the moment the origin be located at the bottom, i.e. at the real source. The mean of the continuity equation becomes

$$\frac{1}{r}\frac{\partial(r\overline{u_r})}{\partial r} + \frac{\partial \overline{u_z}}{\partial z} = 0. \qquad [14]$$

Consider next the vertical component of the momentum balance equation, observing as before, that the buoyancy force density is equal to $mg\rho_{\text{w}}/\rho_{\text{a}}$. We take the mean of the equation, neglect the dynamic pressure $\overline{p_{\text{d}}}$, replace the density ρ of the air–water mixture by ρ_{w}, take into account the continuity equation [14], and assume finally that the magnitude of the vertical derivative of the zz component of the Reynolds stress is much smaller than the derivative of the rz component, i.e. $|\partial/\partial z\overline{u_z'^2}| \ll |\partial/\partial r\overline{u_r'u_z'}|$. The result becomes

$$\frac{\partial}{\partial z}\bar{u}_z^2 + \frac{1}{r}\frac{\partial}{\partial r}(r\bar{u}_r\bar{u}_z) + \frac{1}{r}\frac{\partial}{\partial r}(r\overline{u_r'u_z'}) = \frac{\bar{m}g}{\rho_{\text{a}}}. \qquad [15]$$

We integrate this equation over a fictitious cylinder of infinite width and arbitrary height z, taking into account the boundary conditions

$$r = 0\colon\ \overline{u_r} = 0,\ (r\overline{u_r'u_z'}) = 0,\ \overline{u_z}\ \text{finite},$$

$$r \to \infty\colon\ \overline{u_z} \to 0, (r\overline{u_r'u_z'}) \to 0, (r\overline{u_r})\ \text{finite}.$$

Then

$$\int_0^\infty \bar{u}_z^2 r\,\mathrm{d}r = g\int_0^z \frac{\mathrm{d}z}{\rho_{\text{a}}}\int_0^\infty \bar{m}r\,\mathrm{d}r. \qquad [16]$$

To proceed further we have to model the variations of $\overline{u_z}$, $\bar{m}$, and ρ_{a}. In analogy with [8] and [9] we assume Gaussian distributions:

$$\overline{u_z}(z,r) = u_{\text{c}}(z)\exp\left[-\frac{r^2}{2\sigma^2(z)}\right], \qquad [17]$$

$$\bar{m}(z,r) = m_{\text{c}}(z)\exp\left[-\frac{r^2}{2\lambda^2\sigma^2(z)}\right], \qquad [18]$$

and we assume, as before, that ρ_{a} is determined by the isothermal equation of state for the air. Then [16] yields

$$u_{\text{c}}^2\sigma^2 = \frac{2gP\lambda^2}{\rho_{\text{a}0}}\int_0^z \frac{m_{\text{c}}\sigma^2}{D^* - z}\,\mathrm{d}z, \qquad [19]$$

where $\rho_{\text{a}0}$ is the air density at atmospheric pressure. We take into account the conservation equation for the rate of mass Q_m emitted from the source,

$$Q_m = 2\pi\int_0^\infty \bar{m}(\overline{u_z} + u_{\text{rel}})r\,\mathrm{d}r, \qquad [20]$$

where the relative velocity u_{rel} as before is assumed constant. Upon integration, [20] can be rewritten as

$$m_{\mathrm{c}} = \frac{Q_m}{2\pi\lambda^2\sigma^2[(1+\lambda^2)^{-1}u_{\mathrm{c}} + u_{\mathrm{rel}}]}. \qquad [21]$$

Thus m_{c} can be eliminated from the formalism. Insertion into [19] and differentiation with respect to z yields the momentum equation

$$\frac{\mathrm{d}}{\mathrm{d}z}[u_{\mathrm{c}}^2(z)\sigma^2(z)] = \frac{gQ^0P}{\pi(\mathrm{D}^* - z)[(1+\lambda^2)^{-1}u_{\mathrm{c}}(z) + u_{\mathrm{rel}}]}, \qquad [22]$$

with $Q^0 = Q_m/\rho_{\mathrm{a0}}$. This equation was also obtained by Ditmars & Cederwall (1974). The unknown quantities are u_{c} and σ, whereas λ and u_{rel} are assumed known.

The kinetic energy equation is derived analogously. The mean of the differential equation reads

$$\frac{\partial}{\partial z}\left(\frac{1}{2}\bar{u}_z^3\right) + \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{1}{2}r\bar{u}_r\bar{u}_z^2\right) + \frac{\bar{u}_z}{r}\frac{\partial}{\partial r}(r\overline{u_r'u_z'}) = \frac{\bar{m}\bar{u}_z g}{\rho_{\mathrm{a}}}. \qquad [23]$$

We integrate over the same cylinder volume as before, taking into account the same boundary conditions. After a partial integration of the integral containing the correlation term, and a differentiation of the whole equation with respect to z, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}z}\int_0^\infty \frac{1}{2}\bar{u}_z^3 r\,\mathrm{d}r - \int_0^\infty \overline{u_r'u_z'}\frac{\partial\bar{u}_z}{\partial r}r\,\mathrm{d}r = \frac{g}{\rho_{\mathrm{a}}}\int_0^\infty \bar{m}\bar{u}_z r\,\mathrm{d}r. \qquad [24]$$

Now we insert the self-preservation condition on the correlation $\overline{u_r'u_z'}$, as expressed in [5]. Then using [21], [18], [17] and the isothermal gas law $\rho_{\mathrm{a}} = \rho_{\mathrm{a0}}(D^* - z)/P$, we finally obtain the energy equation in the form

$$\frac{\mathrm{d}}{\mathrm{d}z}[u_{\mathrm{c}}^3(z)\sigma^2(z)] = \frac{3gQ^0P}{\pi(D^* - z)}\frac{u_{\mathrm{c}}(z)}{u_{\mathrm{c}}(z) + (1+\lambda^2)u_{\mathrm{rel}}} - Iu_{\mathrm{c}}^3(z)\sigma(z), \qquad [25]$$

where I is defined in [6]. Since f is not known, the value of I has to be determined by experiment. (The formal analogy between the last terms in [12] and [25] is the reason why it is natural to define I with the prefactor 6.) In view of the self-preservation condition [5] serving as closure relation, the energy equation [25] and the momentum equation [22] form a closed system of equations for the two unknowns u_{c} and σ.

4. METHOD OF SOLUTION

4.1. *Nondimensional form of the equations*

It is convenient to express the governing equations in nondimensional form. We may introduce three nondimensional variables s, ζ, and v, defined by

$$s = \frac{u_{\mathrm{c}}}{u_{\mathrm{rel}}}, \quad \zeta = \frac{z}{D^*}, \quad v = \frac{\pi u_{\mathrm{rel}}u_{\mathrm{c}}^2\sigma^2}{gQ^0P}. \qquad [26]$$

These variables are the three-dimensional analogues of the two-dimensional variables used in Brevik (1977, although with t replaced by s). Equations [22] and [25] can now be written

$$\frac{\mathrm{d}v}{\mathrm{d}\zeta} = \frac{1}{1-\zeta}\frac{1+\lambda^2}{s+1+\lambda^2}, \qquad [27]$$

$$\frac{\mathrm{d}(sv)}{\mathrm{d}\zeta} = \frac{3}{1-\zeta}\frac{s}{s+1+\lambda^2} - Gs^2\sqrt{v}, \qquad [28]$$

where G is a new nondimensional parameter

$$G \equiv \left(\frac{\pi u_{\mathrm{rel}}^3}{gQ^0P}\right)^{1/2} D^*I. \qquad [29]$$

We note that G is the only quantity in the governing equations that describes the strength of the plume source. It is possible to rewrite [28] in the form

$$\frac{\mathrm{d}s}{\mathrm{d}\zeta} = \frac{2-\lambda^2}{1-\zeta}\frac{s}{(s+1+\lambda^2)v} - G\frac{s^2}{\sqrt{v}}. \qquad [30]$$

The two equations [27] and [30] form a closed system for the two unknowns s and v. The system can be integrated with respect to ζ once the initial conditions are known.

4.2. Initial conditions

In the immediate neighbourhood of the real source lying at $z = 0$ the flow depends strongly on the source's geometrical details. It is not until the initial region has been passed, that it becomes possible to speak about an established plume whose characteristics are (practically) independent of the details of the source. According to experimental work of Kobus and others it is meaningful to perform an analytic continuation of the flow field in the established plume down to a virtual point source lying at some position $z = -z_0$. Kobus observed $z_0 = 0.8$ m in his experiment; for the actual water depths this corresponds to $\zeta_0 \simeq 0.05$. Since we shall be concerned mainly with a comparison with Kobus' observations, and also since moderate variations in ζ_0 turn out to have only a slight influence upon the theoretical results, we shall simply assume

$$\zeta_0 = 0.05 \qquad [31]$$

in the following.

For mathematical reasons it will now be convenient to make a translation of the origin: we let

$$z \rightarrow z + z_0, \ \zeta \rightarrow \zeta + \zeta_0, \ D^* \rightarrow D + P + z_0, \qquad [32]$$

whereby the virtual source will be located at $\zeta = 0$.

It turns out to be a quite general property of the various models put forward in the literature that u_c is predicted to be proportional to $z^{-1/3}$ for small z. See, for instance, the comparisons made by Haaland (1979), and the more recent papers of Lemckert & Imberger (1993), and of Islam *et al.* (1995). In view of this it is natural to assume that $u_c \gg u_{rel}$ for small z, whereby the governing equations [27] and [30] can be approximated by

$$\frac{\mathrm{d}v}{\mathrm{d}\zeta} = \frac{1+\lambda^2}{1-\zeta}\frac{1}{s}, \qquad [33]$$

$$\frac{\mathrm{d}s}{\mathrm{d}\zeta} = \frac{2-\lambda^2}{(1-\zeta)v} - G\frac{s^2}{\sqrt{v}}. \qquad [34]$$

The lowest order in ζ, in which case $1-\zeta$ may be replaced by unity in the denominators, [33] and [34] can be solved via die Ansätze $s \propto \zeta^p$, $v \propto \zeta^q$, p and q being constants. The result becomes

$$s = \left[\frac{\sqrt{3}(3-\lambda^2)}{2G\sqrt{1+\lambda^2}}\right]^{2/3} \zeta^{-1/3} \text{ (small } z\text{)}, \qquad [35]$$

$$v = \left[\frac{3G(1+\lambda^2)^2}{4(3-\lambda^2)}\right]^{2/3} \zeta^{4/3} \text{ (small } z\text{)}. \qquad [36]$$

In view of [26] this means $u_c \propto z^{-1/3}$ for small z, in agreement with the other models mentioned above. The standard deviation for $\bar{u}_z$, again in view of [26], becomes

$$\sigma = \left(\frac{gQ^0P}{\pi u_{rel}}\right)^{1/2} \frac{\sqrt{v}}{u_c} \propto \frac{\sqrt{v}}{s} \propto z \text{ (small } z\text{)}, \qquad [37]$$

showing the expected linear variation with z near the virtual source.

Since $s \rightarrow \infty$ when $\zeta \rightarrow 0$, we cannot simply start the integration of [27] and [30] at the origin. One possible solution of this problem would be to employ [35] and [36] a small distance $\Delta\zeta$ away from the origin, and to start the numerical integration of the equations thereafter. However, [35] and [36] are merely lowest order approximations; they quickly become inaccurate some distance away from the origin and an uncritical use of these expressions would imply an appreciable source of error. A simpler and more correct way of approach is to transform to two new variables, henceforth called k and n, that are proportional to ζ near the origin and hence satisfy constant derivative conditions there. The best choice is to take

$$k = sv, \; n = \zeta^{-1/3} v. \qquad [38]$$

The governing equations expressed in terms of these variables near the origin become

$$\frac{\mathrm{d}k}{\mathrm{d}\zeta} = \frac{3}{1-\zeta} - G\frac{k^2}{n^{3/2}}\zeta^{-1/2} \text{ (small } z\text{)}, \qquad [39]$$

$$\frac{\mathrm{d}n}{\mathrm{d}\zeta} = \frac{1+\lambda^2}{1-\zeta}\frac{n}{k} - \frac{n}{3\zeta} \text{ (small } z\text{)}, \qquad [40]$$

and the initial conditions are

$$\zeta = 0\text{: } k = n = 0, \; \frac{\mathrm{d}k}{\mathrm{d}\zeta} = \frac{3}{4}(1+\lambda^2), \; \frac{\mathrm{d}n}{\mathrm{d}\zeta} = \left[\frac{3G(1+\lambda^2)^2}{4(3-\lambda^2)}\right]^{2/3}. \qquad [41]$$

With these initial conditions we can now integrate [39] and [40] upwards from the origin to some position $\zeta = \Delta\zeta$, lying sufficiently close to the origin for the condition $u_c \gg u_{rel}$ to hold. The obtained values of k and n at this position determine in turn via [38] the corresponding values of s and v. The latter values are thereafter used as initial values in the numerical integration of [27] and [30] to give us s and v at an arbitrary nondimensional height ζ. Finally, equations [26] yield the centerline velocity u_c and the standard deviation σ as functions of the height z above the virtual point source.

4.3. *The values of λ and u_{rel}*

The parameter λ is the ratio between the standard deviations for air mass density $\bar{m}$ and vertical water velocity $\bar{u}_z$. As already mentioned, in the literature various proposals have been put forward for the value of this parameter, and the only conclusion that can be drawn with certainty is that λ lies somewhere in the interval between 0 and 1. The limit $\lambda \to 0$ means that all the air rises on the centerline, whereas the limit $\lambda \to 1$ means that there is no motion of water outside the bubble-containing region. Ditmars & Cederwall (1974) chose $\lambda = 0.2$ on the basis of Kobus' data (1968), and the same value was adopted by Brevik (1977). Since λ always occurs in the combination $(1 + \lambda^2)$, the precise value of λ is actually not very important if the inequality $\lambda \ll 1$ holds true. The work of Goossens (1979), however, casts doubts on whether λ really is a small quantity. It is easy to be led astray by visual observations of the plume: a significant part of the air may be contained in the small air bubbles that are subject to larger lateral diffusion than the large dominant bubbles.

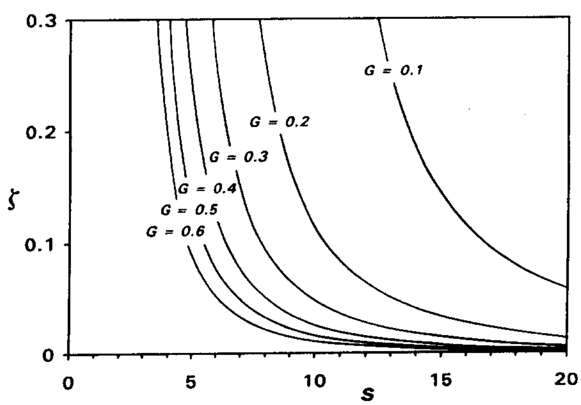

Figure 2. Relation between s and ζ for various values of the parameter G.

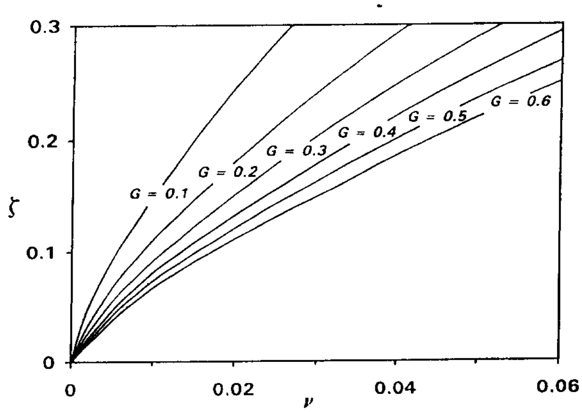

Figure 3. Relation between v and ζ for various values of the parameter G.

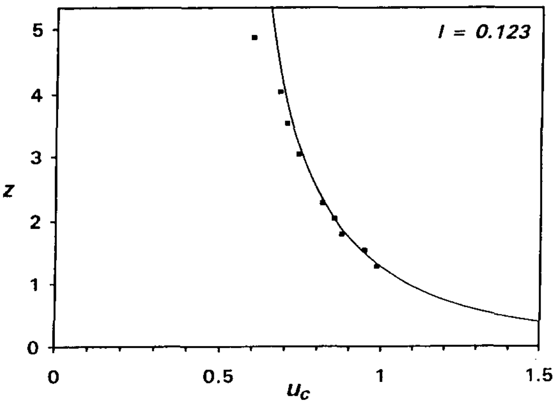

Figure 4. Theoretical centerline velocity u_c (m/s) versus height z(m) above virtual source, when the parameter $I = 0.123$. Air discharge $Q^0 = 2550\ \mathrm{cm^3/s}$. Data points from Kobus (1968).

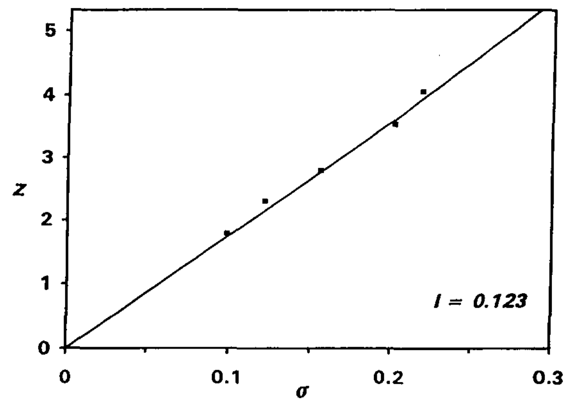

Figure 5. Theoretical standard deviation σ (m) versus height z (m) above virtual source, when $I = 0.123$. Air discharge $Q^0 = 2550\ \mathrm{cm^3/s}$. Data points from Kobus (1968).

As mentioned, Goossens in fact proposed the extreme value $\lambda = 1$, on the basis of his own experiments. The approach adopted by us is more moderate: we shall put

$$\lambda = 0.5 \qquad [42]$$

in the following. This seems to be a reasonable mean among the various proposals put forward in the past.

Consider finally the bubble-water relative velocity u_{rel}. As the value of λ has now been decided upon, it is seen that the governing equations [27] and [30] for s and v contain only one remaining free input parameter, *viz.* G. According to [29], G depends on u_{rel} and I. Therefore, by drawing theoretical curves for $s = s(\zeta)$ and $v = v(\zeta)$ for various values of G, as shown in figures 2 and 3, and by comparing with Kobus' observations for u_c and σ, we can obtain relationships between the magnitudes of u_{rel} and I. Actually it is possible in principle to calculate *both* u_{rel} *and* I explicitly (cf. the more detailed discussion in Brevik 1977). However as this method is somewhat complicated and indirect, it is better to assign a definite, reasonable value to u_{rel} at once, and rather use the formalism's predictions together with Kobus' observations to derive the optimum value of the important parameter I. We shall proceed in this way. We take $u_{rel} = 0.30\ \mathrm{m/s}$, as given already above, in equation [2]. Both Ditmars & Cederwall (1974), Goossens (1979), Haaland (1979), and Milgram (1983) choose values for u_{rel} in the neighbourhood of this one.

5. COMPARISON WITH TWO LARGE SCALE EXPERIMENTS

Our basic theoretical results are those shown in figures 2 and 3. The transform to the dimensional quantities u_c and σ is accomplished via use of [26], as explained at the end of section 4.2.

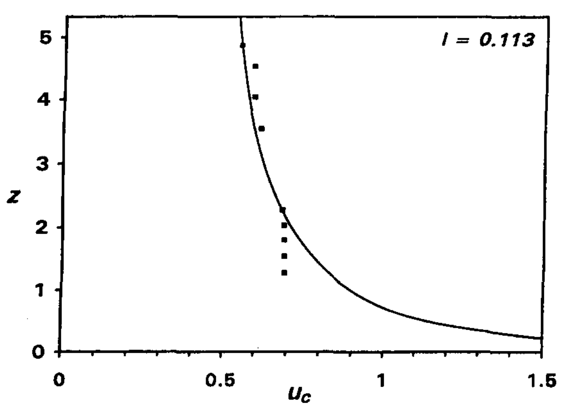

Figure 6. Same as figure 4, but with $Q^0 = 1300\ \mathrm{cm^3/s}$. Optimum parameter choice is $I = 0.113$.

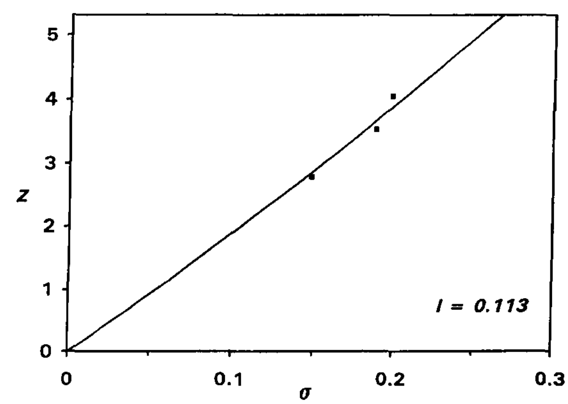

Figure 7. Same as figure 5, but with $Q^0 = 1300\ \mathrm{cm^3/s}$. Optimum parameter choice is $I = 0.113$.

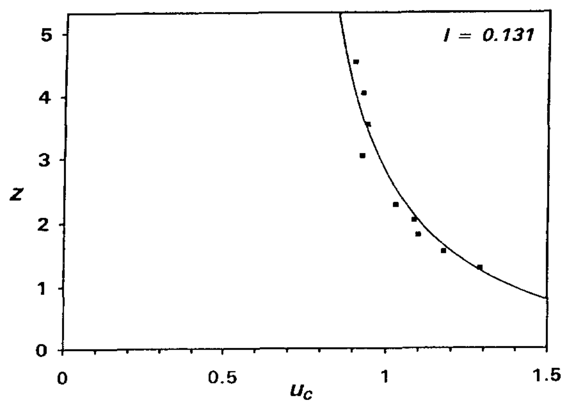

Figure 8. Centerline velocity u_c (m/s) when $Q^0 = 5800\,\text{cm}^3/\text{s}$. Optimum parameter choice is $I = 0.131$.

5.1. Kobus' experiment (1968)

In this experiment the water depth was $D = 4.5$ m. Different air discharges Q^0 were tested. We consider first the case $Q^0 = 2550\,\text{cm}^3/\text{s}$. Theoretical curves can be drawn for $u_c(z)$ and $\sigma(z)$ when various values are chosen for the parameter I. Figures 4 and 5 show the result if I is put equal to 0.123, which turns out to be the optimum value in this case. The agreement with Kobus' experimental points is remarkably good. We do not encounter the problem with excess theoretical values for u_c which was so characteristic in the two-dimensional case. Figures 6 and 7 show, for the same value of D the corresponding results when $Q^0 = 1300\,\text{cm}^3/\text{s}$. The optimum choice for I in this case is $I = 0.113$. Finally, figure 8 shows the centerline velocity when $Q^0 = 5800\,\text{cm}^3/\text{s}$, if I is given the value 0.131. The variations in the optimum values of I in these three cases are thus quite small. It is, moreover, seen that the optimum I for an axisymmetric plume is practically equal to the optimum I_{plane} for a two-dimensional plume; cf. [13]. This coincidence makes it natural to speculate if not the two-dimensional integral [4] and the analogous axisymmetric integral [6] describe after all one and the same physical quantity, having a very universal character.

5.2. Milgram's experiment (1983)

Here, $D = 50$ m. He measured u_c and σ versus z in the case of four different air discharges: $Q^0 = (24\,000,\ 118\,000,\ 283\,000,\ 590\,000)\,\text{cm}^3/\text{s}$. For $Q^0 = 24\,000\,\text{cm}^3/\text{s}$ we get very good agreement with our theoretical model if I is chosen equal to 0.12. Comparisons for u_c and σ are shown in figures 9 and 10, respectively. It is reassuring for the applicability of the theory to note that the optimum value of I is so close to the values of the previous figures. For the larger values of Q^0, it turns out that the optimum values of I become somewhat larger. We shall not go into a further

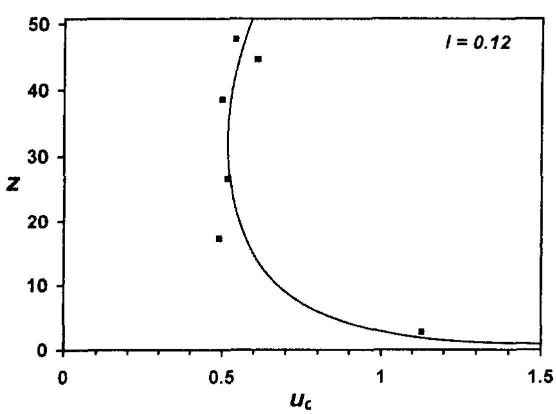

Figure 9. Theoretical centerline velocity u_c (m/s) versus height z (m) above virtual source, when $I = 0.12$. Depth $D = 50$ m, air discharge $Q^0 = 24\,000\,\text{cm}^3/\text{s}$. Data points from Milgram (1983).

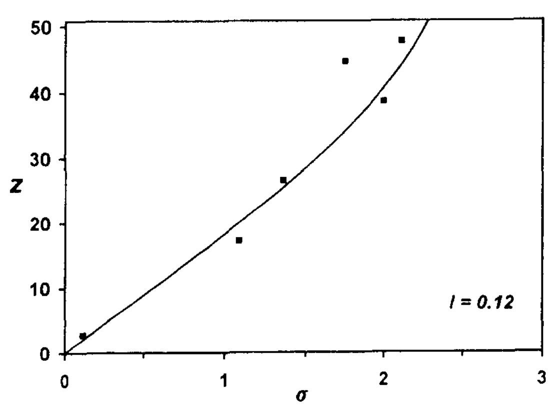

Figure 10. Theoretical standard deviation σ (m) versus height z (m) above virtual source, when $I = 0.12$. Depth $D = 50$ m, air discharge $Q^0 = 24\,000\,\text{cm}^3/\text{s}$. Data points from Milgram (1983).

analysis of this point here, but mention that the four values of Q^0 above correspond to the following optimum values of I (taken in the same order as above): $I = (0.12, 0.15, 0.23, 0.22)$. With these choices for I, the curves for velocities and widths will in all cases turn out to be in quite good agreement with Milgram's observations.

6. REMARKS ON THE EFFECT OF BUBBLE SIZE. CONCLUSIONS

6.1. *On bubble size*

Although we have in this paper had engineering applications in mind, it is of interest to discuss more closely the relation to basic bubble physics, in particular the effects of bubble size.

A general property of large scale bubble plumes seems to be that they give a wide distribution of bubble sizes. The observations show that this distribution depends solely upon the air discharge rate and varies neither with fluid properties nor orifice diameter for all practical cases. The orifice diameter in Kobus' experiment (1968) varied between 0.5 mm and 5 mm. The equivalent bubble diameters d_e were lying between 1 mm and 1 cm. We find it reasonable to assume, however, that this wide range in diameters can for all practical purposes be truncated somewhat: let us assume in the following that the major part of the bubbles had their equivalent diameters lying between 2 mm and 5 mm. It corresponds to Eötvös numbers lying in the region $0.5 \lesssim \mathrm{Eo} \lesssim 4$, where the definition of the Eötvös number is $\mathrm{Eo} = g\Delta\rho d_e^2/\sigma$, $\Delta\rho$ being the water-air mass difference and σ here meaning the surface tension. The single bubble terminal velocity then lies roughly between 20 and 30 cm/s, as already mentioned above (Fig. 7.3 in Clift *et al.*). The single-bubble Reynolds number Re lies in this case roughly in the interval $600 \lesssim \mathrm{Re} \lesssim 1200$. We are then in the so-called intermediate bubble regime; the bubbles show oscillatory trajectories and are more or less ellipsoidal in form. The Eötvös number is in most cases too high to yield purely spherical bubbles, and is too low to enter the spherical-cap region (see, for instance, Fig. 2.5 in Clift *et al.*). As far as the drag coefficient C_D is concerned, we can in cases where $\mathrm{Re} \gtrsim 500$ employ the value $C_D = 0.44$, where C_D is defined relative to the equivalent diameter d_e (Soo 1967, Goossens 1979).

When we instead of one single bubble consider a *swarm* of bubbles, the interaction between the bubbles makes the situation extremely complex. Because the surrounding water is set into vertical motion by the bubbles, the steady-state mean bubble velocity in the plume is greater than the single-bubble velocity considered above. According to Fig. 11 in Kobus (1968) the mean bubble velocity was about 0.7 m/s at $Q^0 = 2000\ \mathrm{cm}^3/\mathrm{s}$, and about 0.9 m/s at $Q^0 = 7000\ \mathrm{cm}^3/\mathrm{s}$. (The weak dependence of the terminal bubble velocity on Q^0 was found to correspond to $(Q^0)^{0.15}$, independent of orifice size as mentioned above.) The flow was subject to considerable fluctuations; Kobus reported that it became necessary to measure over a period of 5 min at each point. The bubbles in such a flow are thus obviously in the intermediate (oscillatory) regime, such as we considered above, but now in a relative sense, i.e. with respect to the ambient water. We recall our assumption $u_{\mathrm{rel}} = 0.30$ m/s for the slip velocity: this is of the same magnitude as the rising velocity of single bubbles in still water. The Reynolds number for the bubbles in the plume, relative to the ambient water, should then for $2\ \mathrm{mm} \lesssim d_e \lesssim 5\ \mathrm{mm}$ be expected to lie in the interval $600 \lesssim \mathrm{Re}_{\mathrm{rel}} \lesssim 1500$. The occurrence of strong oscillations in a plume of this kind is just as we would expect.

The magnitude of the Reynolds number is of importance for the following two processes:

(i) the interaction between bubbles; and

(ii) the turbulence and wakes generated by the bubbles.

We note in this context that in any bubble plume there are two primary length scales associated with the production of turbulence, namely the width of the plume as in single phase flow, and the bubble radii (cf. for instance, the discussion of Sjøen 1983). The turbulence intensity in the plume can thus be considered to be the result of the fluctuations produced by the turbulent shear flow of the main plume itself, and the fluctuations produced by the small scale turbulent shear flow produced by the wakes of the bubbles. A typical frequency of the production of turbulent eddies in the wakes is obtained by dividing the relative velocity with an average bubble radius. This gives a frequency of about 100 Hz. A similar estimate of the frequency for large scale eddies of the plume is obtained by dividing the plume velocity (~ 1 m/s) by the plume width (~ 1 m), i.e. a frequency of about 1 Hz. The characteristic frequency for the eddies of the main plume is thus about two orders of magnitude lower than the characteristic frequency of the bubble-generated fluctuations.

This is typical for large scale bubble plumes. It is reasonable to assume that it is the lower frequencies that contain most of the energy. The bubble-generated turbulence (item (ii) above) in large plumes is expected to be relatively small. As regards the interaction between the bubbles, we expect these effects to be strong at the actual Reynolds numbers because of the instabilities and the strong mixing. One should expect nonlinear effects to play an important role here. The bubble-interaction problem is difficult to model analytically; one has to rely upon experimental observations.

We wish to stress finally that there may occur significant new properties of bubble plumes when their dimensions become very large. In Bulson's experiment (1961), where the depth was as large as 11 m, and the flow rates per unit length were high, up to $Q^0 = 1500\,\text{cm}^2/\text{s}$, there thus occurred a core region at the plume centre which was largely gaseous. Such a core will reduce the resistance to bubble rise. Cf. also the discussion by Wilkinson (1979). In this case the average bubble size in the core was probably quite large. Even more extreme conditions should be expected in the $D = 50\,\text{m}$ experiment of Milgram (1983).

6.2. Conclusions, and further remarks

Let us summarize as follows:

(1) A central element in the present theory is the use of the kinetic energy equation, which takes the form [25] after the imposition of the self-preservation property [5] for the most dominant Reynolds stress component. When this equation is taken together with the momentum equation [22], we obtain the two governing equations for the assumed continuous air-water system, given by [27] and [30] in nondimensional form. The results are shown in figures 2 and 3. Reinstatement of dimensional units is made via [26], permitting us to calculate u_c and σ versus height z above the virtual point source once the input parameters are known.

(2) The parameter λ, defined by [18], is put equal to 0.5, the slip velocity u_{rel} is put equal to 0.30 m/s, and the nondimensional displacement of the virtual source $\zeta_0 = z_0/D^*$ is put equal to 0.05. These are fixed parameters. There remains one single nondimensional parameter in the formalism, namely the integral I defined in [6], whose value has to be determined from experiments. Comparison with Kobus' experiment (1968) makes it reasonable to expect that the value

$$I = 0.12 \qquad [43]$$

is quite appropriate under moderate large scale circumstances. The deviations from this value with use of different air discharges Q^0 in Kobus' experiment were found to be small. Moreover, [43] practically agrees with the value of the analogous quantity I_{plane} in the case of two-dimensional flow; cf. [13]. In practice, for a given large scale plume with known values of D and Q^0, we can easily calculate G from [29] using [43] and the adopted value $u_{rel} = 0.30$ m/s, and so the nondimensional solution can immediately be read off from figures 2 and 3.

However, in extreme cases of very large depths and air discharges, there are indications that the optimum values of I increase somewhat. Thus in the $D = 50$ m experiment of Milgram (1983), in which Q^0 took values as large as about $600\,000\,\text{cm}^3/\text{s}$, it turned out that the optimum value of I was lying around 0.2. In practice, one therefore has to know the actual air discharge, at least roughly, before applying the theory.

It should also be observed that the optimum values for λ and I are interrelated. We actually tested our theoretical model against Milgram's experiment choosing $\lambda = 0.8$ as input (this was the value for λ adopted by him). It turned out to be possible to obtain the same kind of agreement as above, with slightly smaller optimum values of I. For instance, the case $Q^0 = 24\,000\,\text{cm}^3/\text{s}$ was for $\lambda = 0.8$ found to correspond to $I = 0.08$. Therefore, as far as the very large scale experiment of Milgram is concerned, there seems to be no reason for preferring one particular value of λ instead of another. On this very large scale, the important point is rather that of mutual consistency in the parameter choices.

(3) As figures 4–10 show, the agreement between theory and experiment in the axisymmetric case is good. There is no sign of the problem with excessive theoretical values for u_c which was so characteristic for the analogous theory in the two-dimensional case (Brevik 1977). The

physical reason for this may be that the three-dimensional nature of air bubbles generally makes an axisymmetric model of a single-orifice plume more adaptable than a two-dimensional model of a multiply-orifice plume. An important point is also how large is the *spacing* between the orifices in the pipeline; in Kobus' experiment the spacing was quite large (10 cm). The larger the spacing, the less adaptable one expects the two-dimensional model to be.

(4) There exists an experiment of Topham (1975) with air bubble plumes using orifice depths up to 60 m and air discharge rates up to 660 000 $\mathrm{cm^3/s}$. The results for plume widths and centerline velocities versus height show, however, considerable deviations from smooth functions, and this experiment is not further analysed here. The reader is referred to the discussion of Milgram (1983) on this point. The reason for the deviations in the measured results may have been turbulence-induced disturbances in the measuring equipment.

(5) How large is the influence from *turbulence* in the flow field? This point is interrelated to the value of λ. Goossens (1979), for instance, introduced the idea of an equivalent velocity profile for bubble plumes, claiming that in most of the previous works there has been an underestimate of turbulence through the omission of relevant Reynolds stress terms in the governing equations. Thus the presence of excess centerline velocities in two-dimensional models, such as in those of Ditmars & Cederwall (1974) and Brevik (1977), is in his opinion ascribed to this effect.

The key question is whether the correlation $\overline{u_z'^2}$ is really negligible. This correlation is of importance for the vertical momentum flux. From the analogous case of a single phase free jet, it is known that the vertical correlation amounts only by about 10% to the momentum flux and can so be disregarded. In view of this, it might perhaps appear to be most natural to neglect the correlation when modelling bubble plumes also, such as we have done above, in agreement with most other workers. However, the bubble plume is after all a two-phase system, and some caution is called for when comparing with single phase systems. Goossens made use of his observed data in an attempt to evaluate how large a part of the total momentum flux was actually connected with the term $\overline{u_z'^2}$. The result was reported to be quite appreciable, typically around 30% or more.

However, we have to stress here that Goossens' analysis is interwoven with his assumption of a very large value of λ. He assumed $\lambda = 1$. We actually made an explicit test of how a large λ input value fits into our theoretical model, under moderate large scale conditions. Specifically, for the case $D = 4.50$ m, $Q^0 = 2550\ \mathrm{cm^3/s}$ we repeated our calculation of u_c and σ inserting $\lambda = 0.9$ or $\lambda = 1$ as input. We found that the experimental results could not be reproduced simultaneously for *any* common value of I. What is clear, is that under moderate large scale conditions, such large values of λ do *not* fit into our formalism in a natural way. Altogether, when considering Kobus' and Milgram's experiment in combination, we find it most appropriate to choose $\lambda = 0.5$ as the most optimum general value. It is conceivable that this parameter value indirectly takes into account some part of the turbulence effect also. For the convenience of the reader, table 1 summarizes some optimum parameter values that we recommend.

Table 1. Proposed optimum parameter values in order to fit our theoretical model with experiment. It is assumed that $\lambda = 0.5$, $u_{rel} = 0.30$ m/s throughout. Water depth $D = 4.5$ m corresponds to Kobus (1968), whereas $D = 50$ m corresponds to Milgram (1983)

D (m)	Q^0 ($\mathrm{cm^3/s}$)	I
4.5	1300	0.113
4.5	2550	0.123
4.5	5800	0.131
50	24 000	0.12
50	118 000	0.15
50	283 000	0.23
50	590 000	0.22

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